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On strongly closed subgraphs with diameter two and the Q-polynomial property

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Abstract

In this paper, we study a distance-regular graph $\Gamma = (X, R)$ with an intersection number $a_2 \neq 0$ having a strongly closed subgraph Y of diameter 2. Let $E_0, E_1, \ldots, E_D$ be the primitive idempotents corresponding to the eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$ of Γ. Let $V = \boldsymbol{C}^X$ be the vector space consisting of column vectors whose rows are labeled with the vertex set X. Let W be the subspace of V consisting of vectors whose supports lie in Y. A nonzero vector $\boldsymbol{v} \in W$ is said to be tight whenever $E_0\boldsymbol{v}$ and at least one of $E_1\boldsymbol{v}, \ldots, E_D\boldsymbol{v}$ is zero. We show that the existence of a tight vector in W is equivalent to a balanced condition defined by P. Terwilliger. As an application, we study the structure of parallelogram-free distance-regular graphs and conditions for these graphs to be Q-polynomial.

1. Introduction

Let $\Gamma = (X, R)$ be a distance-regular graph of diameter D with vertex set X and edge set R. For vertices x and y, $\partial(x, y)$ denotes the distance between x and y, i.e., the length of a shortest path connecting x and y. For a vertex $u \in X$ and $j \in \{0, 1, \ldots, D\}$, let

$$\Gamma_j(u) = \{x \in X \mid \partial(u, x) = j\} \qquad \text{and} \qquad \Gamma(u) = \Gamma_1(u).$$

For two vertices u and $v \in X$ with $\partial(u, v) = j$ let

$$C(u, v) = \Gamma_{j-1}(u) \cap \Gamma(v),$$

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$$A(u, v) = \Gamma_j(u) \cap \Gamma(v), \quad \text{and}$$
$$B(u, v) = \Gamma_{j+1}(u) \cap \Gamma(v).$$

The cardinalities $c_j = |C(u, v)|$, $a_j = |A(u, v)|$ and $b_j = |B(u, v)|$ depend only on $j = \partial(u, v)$, and they are called the *intersection numbers* of Γ. Let $k_i = |\Gamma_i(u)|$ and $k = k_1$. Then $k_0 = 1$,

$$b_i k_i = c_{i+1} k_{i+1} \qquad \text{for } i = 0, 1, \ldots, D-1, \tag{1}$$

and k_i does not depend on the choice of u. The number k_i is called the ith valency, and k the valency of Γ.

A subset Y of the vertex set X is said to be *strongly closed* if the following condition is satisfied:

$$C(u, v) \cup A(u, v) \subset Y \qquad \text{for all } u, v \in Y.$$

We often identify a subset of X with the induced subgraph on it. In particular, when Y is strongly closed, Y is referred to as a strongly closed subgraph of Γ.

Strongly closed subgraphs were defined in [10]. Many distance-regular graphs have strongly closed subgraphs and conditions for guaranteeing the existence of such subgraphs have been studied, for example, in [3–6,11,18,19]. In this paper, we study the connection between the eigenvalues of a strongly closed subgraph and those of Γ. In order to state our results, we make a few more definitions.

Let $V = \boldsymbol{C}^X$ denote the vector space over the complex number field consisting of column vectors whose coordinates are indexed by X with complex entries. For all $x \in X$, let $\hat{x}$ denote the element of V with a 1 in the x-coordinate and 0 in all other coordinates. For a vector $\boldsymbol{v} = \sum_{x \in X} \alpha(x)\hat{x} \in V$ expressed as a linear combination of $\hat{x}$'s, supp($\boldsymbol{v}$) denotes the *support* of $\boldsymbol{v}$, i.e., $\mathrm{supp}(\boldsymbol{v}) = \{x \in X \mid \alpha(x) \neq 0\}$.

Let Y be a subset of X. Let $E^* = E^*(Y)$ denote the projection onto the subspace spanned by vectors $\hat{y}$ with $y \in Y$.

We now state our main results.

Theorem 1.1. *Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geq 3$, and an intersection number $a_2 > 0$. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ be eigenvalues of Γ, i.e., those of the adjacency matrix A of Γ, and E_i the primitive idempotent corresponding to θ_i for $i = 0, 1, \ldots, D$. Let Y be a strongly closed regular subgraph of Γ of diameter two. Then the following hold.*

(i) *Y is strongly regular, i.e., distance-regular of diameter two, with eigenvalues η_0, η_1 and η_2 satisfying $\eta_0 = c_2 + a_2 > \eta_1 > 0 > -1 > \eta_2$, and*

$$\theta_1 \leq -1 - \frac{b_1}{1 + \eta_2}, \qquad \theta_D \geq -1 - \frac{b_1}{1 + \eta_1}. \tag{2}$$

(ii) *If there is a nonzero vector $\boldsymbol{v} \in E^*V$ satisfying $E_0\boldsymbol{v} = \boldsymbol{0}$ and $E_i\boldsymbol{v} = 0$ for some $i \in \{1, 2, \ldots, D\}$, then $i = 1$ or D.*

(iii) *Let $i = 1$ or D. Set $\eta = \eta_2$ if $i = 1$, and $\eta = \eta_1$ if $i = D$. Then the following are equivalent.*

(a) $\theta_i = -1 - b_1/(1 + \eta)$.

(b) *There is a nonzero vector* $\boldsymbol{v} \in E^*V$ *such that* $E_0\boldsymbol{v} = E_i\boldsymbol{v} = \boldsymbol{0}$.
(c) $E_0\boldsymbol{v} = E_i\boldsymbol{v} = \boldsymbol{0}$ *for every vector* $\boldsymbol{v} \in E^*V$ *satisfying* $E^*A\boldsymbol{v} = \eta\boldsymbol{v}$.
(d) *For every* $x, y \in Y$ *with* $\partial(x, y) = 2$, $E_0\boldsymbol{u} = E_i\boldsymbol{u} = \boldsymbol{0}$, *where*

$$\boldsymbol{u} = \sum_{z\in A(y,x)} \hat{z} - \sum_{w\in A(x,y)} \hat{w} - \eta(\hat{x} - \hat{y}).$$

The condition in (iii)(d) above is a balanced condition defined by Terwilliger in [13,14]. Balanced conditions are closely related to the Q-polynomial property of distance-regular graphs.

Recently, in [16], Terwilliger and Weng showed that if θ_1 is the second largest eigenvalue of a regular near polygon with diameter $D \geq 3$, valency k and intersection numbers $a_1 > 0$, $c_2 > 1$, then

$$\theta_1 \leq \frac{k - a_1 - c_2}{c_2 - 1}. \tag{3}$$

Equality is attained above if and only if Γ is Q-polynomial with classical parameters with respect to θ_1.

In this case Γ contains a strongly closed strongly regular subgraph with valency $c_2(1 + a_1)$ and the least eigenvalue is $-c_2$. Hence the first inequality in (2) is nothing but the inequality (3).

The second result in this paper is a characterization of the Q-polynomial property of parallelogram-free distance-regular graphs. Recall that a *parallelogram* of length $j+1 \geq 2$ is a four-vertex configuration (w, x, y, z) such that $\partial(w, x) = \partial(y, z) = j = \partial(x, z)$, $\partial(x, y) = \partial(z, w) = 1$ and $\partial(w, y) = j + 1$. Regular near polygons do not have parallelograms of any lengths. There is a series of excellent articles on parallelogram-free distance-regular graphs by C. Weng and others. See [9,15,17–20].

Theorem 1.2. *Let* $\Gamma = (X, R)$ *be a parallelogram-free distance-regular graph with diameter* $D \geq 3$, *and intersection numbers* $a_2 > 0$ *and* $b_1 > b_2$. *Let* $\theta_0 > \theta_1 > \cdots > \theta_D$ *be the eigenvalues of* Γ. *Then the roots* $\eta_1 \geq \eta_2$ *of the quadratic equation* $x^2 + (c_2 - a_1)x - a_2$ *satisfy* $\eta_1 > 0 > -1 > \eta_2$, *and the following hold.*

(i) $\theta_1 \leq -1 - \frac{b_1}{1+\eta_2}$, *and* $\theta_D \geq -1 - \frac{b_1}{1+\eta_1}$.
(ii) *Suppose* $\theta \in \{\theta_1, \theta_D\}$ *attains one of the bounds above. Let* $q = b_1/(\theta + 1)$. *Then*
  (a) *The intersection numbers of* Γ *are such that*

$$qc_i - b_i - q(qc_{i-1} - b_{i-1}) \text{ is independent of } i \ (1 \leq i \leq D).$$

  (b) $c_3 \geq (c_2 - q)(q^2 + q + 1)$.
  (c) *If* $\theta = \theta_1$, *then* $q + 1 \geq c_2$ *and* $q^2 + q + 1 \geq c_3$, *and if* $\theta = \theta_D$, *then* $q + 1 \leq -a_1$.
  (d) *The equality holds in* (b) *if and only if* Γ *is* Q*-polynomial with classical parameters* (D, q, α, β) *with suitable choices of real numbers* α *and* β.

If Γ is a regular near polygon of diameter $D \geq 3$, then it is parallelogram-free and $c_2a_1 = a_2$. So $\eta_2 = -c_2$. Hence if θ_1 attains the bound in (i), then $q = c_2 - 1$. Now (b), (c) and (d) in (ii) imply that Γ is Q-polynomial with classical parameters.

2. Preliminaries

In this section we recall some facts about distance-regular graphs. For the general theory of distance-regular graphs, we refer the reader to [1,2].

Let X denote a nonempty finite set. Let $\mathrm{Mat}_X(\boldsymbol{C})$ denote the complex algebra consisting of all matrices whose rows and columns are indexed with X with complex entries. Let $V = \boldsymbol{C}^X$ denote the vector space over the complex number field consisting of column vectors whose coordinates are indexed by X with complex entries. We observe $\mathrm{Mat}_X(\boldsymbol{C})$ acts on V by left multiplication. We endow V with the Hermitian inner product $\langle\ ,\ \rangle$ defined by

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = {}^t\boldsymbol{u}\bar{\boldsymbol{v}} \quad (\boldsymbol{u}, \boldsymbol{v} \in V),$$

where ${}^t\boldsymbol{u}$ denotes the transpose of $\boldsymbol{u}$, and $\bar{\boldsymbol{v}}$ denotes the complex conjugate of $\boldsymbol{v}$. We use the abbreviation $\|\boldsymbol{u}\|^2 = \langle \boldsymbol{u}, \boldsymbol{u}\rangle$ for all $\boldsymbol{u} \in V$.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set R. For $x, y \in X$, let $\partial(x, y)$ denote the *distance* between x and y, that is the length of a shortest path connecting x and y. The *diameter* D is the maximal distance between vertices. The graph Γ is said to be *distance-regular* whenever for all integers $h, i, j \in \{0, 1, \ldots, D\}$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{i,j} = |\{z \in X \mid \partial(x, z) = i, \partial(z, y) = j\}| \tag{4}$$

is independent of x and y. We make the abbreviations $c_i = p^i_{i-1,1} (1 \le i \le D)$, $a_i = p^i_{i,1} (0 \le i \le D)$, $b_i = p^i_{i+1,1} (0 \le i \le D-1)$ and $k = b_0$. For notational convenience, we define $c_0 = 0$ and $b_D = 0$. Γ is regular of valency k and we have

$$k = c_i + a_i + b_i \qquad \text{for all } i = 0, 1, \ldots, D. \tag{5}$$

A *strongly regular* graph, in this paper, is a distance-regular graph of diameter 2.

For the rest of this paper we assume Γ is distance-regular with diameter D.

For $i \in \{0, 1, \ldots, D\}$ let A_i denote the matrix in $\mathrm{Mat}_X(\boldsymbol{C})$ whose (x, y)-entry is defined by

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix A_i is called the *ith adjacency matrix* of Γ. For $i, j \in \{0, 1, \ldots, D\}$ we have

$$A_i A_j = \sum_{h=0}^{D} p^h_{i,j} A_h. \tag{6}$$

In particular, using triangular inequalities we have for $i \in \{1, \ldots, D\}$

$$A_i A_1 = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \tag{7}$$

by setting $A_{D+1} = 0$ and $c_{D+1} = 1$. Observe that by (6) and (7), the linear span $\mathcal{M} = \mathrm{Span}(A_0, A_1, \ldots, A_D)$ is closed under multiplication and it is algebraically generated by $A = A_1$. $\mathcal{M}$ is called the *Bose–Mesner algebra* of Γ. Since $\mathcal{M}$ is commutative and generated by real symmetric matrices, it has a basis consisting of primitive idempotents. Let $E_0, E_1, E_2, \ldots, E_D$ be the primitive idempotents. We write

$$A_i = \sum_{j=0}^{D} p_i(j) E_j \qquad \text{for all } i \in \{0, 1, \ldots, D\}, \tag{8}$$

and

$$E_i = \frac{1}{|X|} \sum_{j=0}^{D} q_i(j) A_j \qquad \text{for all } i \in \{0, 1, \ldots, D\}. \tag{9}$$

Set $m_i = q_i(0)$ and $\theta_i = p_1(i)$. Then $\theta_0, \theta_1, \ldots, \theta_D$ are the distinct eigenvalues of $A = A_1$, and m_i is the multiplicity of θ_i in A. We order $E_0, E_1, \ldots, E_D$ so that

$$\theta_0 > \theta_1 > \cdots > \theta_D.$$

Since Γ is a connected k-regular graph, its adjacency matrix has largest eigenvalue k with multiplicity 1. Hence $E_0 = \frac{1}{|X|} J$ in this ordering, where $J \in \mathrm{Mat}_X(\boldsymbol{C})$ is the all 1's matrix. We use the following well-known formulas. For all $i, j \in \{0, 1, \ldots, D\}$,

$$\frac{p_i(j)}{k_i} = \frac{q_j(i)}{m_j}. \tag{10}$$

Let $v_0(t), v_1(t), \ldots, v_D(t), v_{D+1}(t)$ denote polynomials in $\boldsymbol{R}[t]$ satisfying $v_0(t) = 1$ and for $i \in \{0, 1, \ldots, D\}$,

$$t v_i(t) = b_{i-1} v_{i-1}(t) + a_i v_i(t) + c_{i+1} v_{i+1}(t), \tag{11}$$

where $b_{-1} = 0$, $c_{D+1} = 1$ and $v_{-1}(t) = 0$. Then for each integer $i \in \{0, 1, \ldots, D+1\}$, the polynomial $v_i(t)$ has degree i, and the leading coefficient $(c_1 c_2 \cdots c_i)^{-1}$. Moreover, by (7), we have $v_i(A_1) = A_i$ with $A_{D+1} = O$. Hence $p_i(j) = v_i(\theta_j)$ by (8). For each eigenvalue θ_j, let

$$\sigma_i = \sigma_i(\theta_j) = \frac{q_j(i)}{m_j} = \frac{p_i(j)}{k_i}.$$

The numbers $\sigma_0, \sigma_1, \ldots, \sigma_D$ are called the *cosine sequence* associated with θ_j. Let $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote the cosine sequence associated with $k = \theta_0$. Then $\sigma_i = 1 (0 \le i \le D)$. By the *trivial cosine sequence* of Γ we mean the cosine sequence associated with k.

Lemma 2.1. *Let Γ be a distance-regular graph of diameter D. Let E be a primitive idempotent associated with the eigenvalue θ, and let $\sigma_0, \sigma_1, \ldots, \sigma_D$ be its cosine sequence. Let σ_{-1} and σ_{D+1} be indeterminates. Then the following hold.*

(i) $c_i \sigma_{i-1} + a_i \sigma_i + b_i \sigma_{i+1} = \theta \sigma_i$ *for* $i = 0, 1, \ldots, D$.
(ii) $c_i(\sigma_{i-1} - \sigma_i) + b_i(\sigma_{i+1} - \sigma_i) = (\theta - k)\sigma_i$ *for* $i = 0, 1, \ldots, D$.
(iii) $\sigma_0 = 1$, $\sigma_1 = \theta/k$,

$$\sigma_2 = \frac{\theta^2 - a_1\theta - k}{k b_1}, \qquad \text{and}$$

$$\sigma_3 = \frac{\theta^3 - (a_1 + a_2)\theta^2 + (a_1 a_2 - k - c_2 b_1)\theta + a_2 k}{k b_1 b_2}.$$

(iv) $\sigma_0 - \sigma_1 = (k-\theta)/k$,

$$\sigma_0 - \sigma_2 = \frac{1}{kb_1}(k-\theta)(\theta+1+b_1),$$
$$\sigma_0 - \sigma_3 = \frac{1}{kb_1b_2}(k-\theta)(\theta^2+(k-a_1-a_2)\theta+b_1b_2-a_2),$$
$$\sigma_1 - \sigma_2 = \frac{1}{kb_1}(k-\theta)(\theta+1),$$
$$\sigma_1 - \sigma_3 = \frac{1}{kb_1b_2}(k-\theta)(\theta^2+(k-a_1-a_2)\theta-a_2), \qquad \textit{and}$$
$$\sigma_2 - \sigma_3 = \frac{1}{kb_1b_2}(k-\theta)(\theta^2+(c_2-a_1)\theta+c_2-k).$$

Proof. (i) follows from (10), (11) and the definition of σ_i using (1). Since $k = c_i + a_i + b_i$, we have (ii). The remaining follow from (i) and (ii). Most of the formulas above can be found in [1,2,8]. □

Lemma 2.2 *([8, Lemma 2.6]). Let Γ denote a distance-regular graph with diameter $D \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$. Then the following hold.*

(i) $k > \theta_1 > 0$.
(ii) $-1 > \theta_D \geq -1 - b_1$.
(iii) *Suppose Γ is not bipartite. Then $\theta_D > -1 - b_1$.*

3. Existence of strongly closed subgraphs

In this section we review the results on the existence of strongly closed subgraphs. In this paper, we need only the case when $b_1 > b_2$ and $a_2 \neq 0$. For more general cases, see [3–6]. We first recall two conditions defined in [4].

Definition 3.1. Let $\Gamma = (X, R)$ be a distance-regular graph of diameter D. Let j be an integer $1 \leq j \leq D-1$.

SC_j: For every pair of vertices x and y with $\partial(x,y) = j$, there is a strongly closed subgraph containing x and y of diameter j.

SS_j: For all vertices x, y and $z \in X$ such that $\partial(x,z) = \partial(y,z) = j$ and $\partial(x,y) = 1$, $B(x,z) = B(y,z)$.

Note that the condition SS_j is nothing but the nonexistence of a parallelogram of length $j+1$.

Proposition 3.1 *([19, Theorem 1], [11, Theorem 1.1]). Let $\Gamma = (X, R)$ be a distance-regular graph of diameter $D \geq 3$. Suppose $b_1 > b_2$.*

(i) *If $a_1 \neq 0$, then for a positive integer $i < D$, the following conditions are equivalent.*
  (a) *SC_j holds for every $j \in \{1, 2, \ldots, i\}$.*
  (b) *SS_j holds for every $j \in \{1, 2, \ldots, i\}$.*

(ii) *If $a_2 \neq 0$, then the condition SC_2 holds if and only if the conditions SS_1 and SS_2 hold.*

Moreover, if the conditions are satisfied, then all strongly closed subgraphs guaranteed to exist by condition SC_j in (i)(a) *or SC_2 in* (ii) *are distance-regular.*

Proposition 3.2. *Let $\Gamma = (X, R)$ be a distance-regular graph with diameter D. Let ℓ be an integer $2 \leq \ell \leq D$. Suppose Γ satisfies the conditions SS_m for $m = 1, 2, \ldots, \ell - 1$. Let h, i and j be integers such that $h \geq 0$, $1 \leq j \leq i \leq \ell - h$. If u, v, w, x are vertices of Γ such that*

$$\partial(x, v) = h + i, \partial(x, u) = h, \partial(u, v) = i, \partial(u, w) = j, \partial(w, v) = i - j + 1,$$

then $\partial(x, w) = h + j$.

Proof. We first prove a special case $i = j$ by induction on h. There is nothing to prove if $h = 0$. Let $h \geq 1$ and $y \in C(u, x)$. Then by the induction hypothesis $\partial(y, w) = h + i - 1$. Since $\partial(y, v) = h + i - 1$, the condition SS_{h+i-1} forces $\partial(x, w) = h + i$ as $x \in B(v, y) = B(w, y)$. This proves the special case.

We now prove the general case by induction on h. There is nothing to prove if $h = 0$. Let $h \geq 1$ and $y \in C(u, x)$. Suppose $\partial(x, w) \neq h + j$. Then by a triangular inequality, $\partial(x, w) = h + j - 1$. By the induction hypothesis, $\partial(y, w) = h + j - 1$. Let $h' = i - j + 1$, $i' = j' = h + j - 1$, $x' = v$, $u' = w$, $v' = x$ and $w' = y$. Then by the special case we treated in the previous paragraph, $\partial(y, v) = h + i$. This is a contradiction as

$$\partial(y, v) \leq \partial(y, u) + \partial(u, v) = h - 1 + i.$$

This proves the assertion. □

Recall that a kite of length $i \geq 2$ is a four-vertex configuration (w, x, y, z) such that $\partial(w, x) = \partial(w, z) = i - 1$, $\partial(x, y) = \partial(y, z) = \partial(z, x) = 1$ and $\partial(w, y) = i$.

Corollary 3.3. *Let $\Gamma = (X, R)$ be a distance-regular graph with diameter D. Let ℓ be an integer $2 \leq \ell \leq D$. Suppose Γ satisfies the conditions SS_j for $j = 1, 2, \ldots, \ell - 1$. Let i be a positive integer such that $i \leq \ell$, and let u, v, x be vertices of Γ such that*

$$\partial(x, v) = \ell, \qquad \partial(x, u) = \ell - i \qquad \text{and} \qquad \partial(u, v) = i.$$

Then $A(u, v) \subset \Gamma_\ell(x)$ and $A(v, u) \subset \Gamma_{\ell-i+1}(x)$. In particular, Γ does not have a kite of length ℓ.

Proof. The assertions follow directly from Proposition 3.2 by setting $j = i$ or $j = 1$.
To show that there is no kite, set $i = 1$. □

4. Strongly closed subgraph of diameter 2

In this section, we prove Theorem 1.1. Throughout this section, we assume the following.

Hypothesis 4.1. Let $\Gamma = (X, R)$ be a distance-regular graph of diameter $D \geq 3$ with $a_2 \neq 0$. Let Y be a strongly closed subset of X. Suppose the induced subgraph on Y is regular with diameter 2, and Y does not contain a parallelogram of length 2.

Let $\Delta = (Y, R_{|Y\times Y})$ be the subgraph of Γ induced on Y. Since Δ is strongly closed and regular, it is connected and strongly regular of valency $\kappa = c_2 + a_2$. Set $\lambda = a_1$ and $\mu = c_2$. Since Δ does not contain a parallelogram of length 2, $0 \neq a_2 \geq c_2 a_1$. For a strongly regular graph we adopt the notation $(\nu, \kappa, \lambda, \mu)$ to describe its parameters; here ν is the number of vertices. Recall that a conference graph is a strongly regular graph with parameters $(4\mu + 1, 2\mu, \mu - 1, \mu)$.

Lemma 4.1. *Let $\Delta = (Y, R_{|Y\times Y})$ be the subgraph of Γ induced on Y. Then Δ is strongly regular with parameters $(\nu, \kappa, \lambda, \mu)$ with $\kappa = c_2 + a_2$, $\lambda = a_1$ and $\mu = c_2$ and the following hold.*

(i) $0 \neq \kappa - \mu \geq \lambda\mu$.
(ii) *The eigenvalues of Δ are $\eta_0 > \eta_1 > \eta_2$ with $\eta_0 = \kappa$, $\eta_1 > 0$, $\eta_2 < -1$, where η_1 and η_2 are the roots of the equation*

$$t^2 + (\mu - \lambda)t + (\mu - \kappa) = 0.$$

(iii) *All eigenvalues of Δ are integers if $\kappa > 2$.*
(iv) *Let $\theta_0 > \theta_1 > \cdots > \theta_D$ be the distinct eigenvalues of Γ. Then*

$$\theta_1 \geq \eta_1 > 0, \quad -1 > \eta_2 \geq \theta_D.$$

Proof. See [2, Theorem 1.3.1]. For integrality of eigenvalues in (iii), it suffices to consider the case when Δ is a conference graph. Since $\lambda\mu \leq \kappa - \mu = \mu$, either $\lambda = 1$ or 0. If $\lambda = 1$, the eigenvalues are integral. If $\lambda = 0$, then $\kappa = 2$, which is excluded. (iv) follows from Corollary 3.3.2 in [2] using (ii). $\square$

Let $E_i^* = E_i^*(Y)(i = 0, 1, \ldots, D)$ denote the diagonal matrices in $\mathrm{Mat}_X(\boldsymbol{C})$ defined by the following.

$$(E_i^*)_{x,y} = \begin{cases} 1 & \text{if } x = y \text{ and } \partial(x, Y) = i, \\ 0 & \text{otherwise.} \end{cases}$$

In [12] we defined the subalgebra $\mathcal{T} = \mathcal{T}(Y)$ of $\mathrm{Mat}_X(\boldsymbol{C})$ generated by A and $E_0^*, E_1^*, \ldots, E_D^*$. We review a couple of results in [12]. Since we need only E_0^* in this paper, write $E^* = E_0^*$ and $W = E^*V$, where $V = \boldsymbol{C}^X$. Set $\widetilde{A} = E^*AE^*$. Then W is the vector subspace of V consisting of the vectors whose supports are in Y. Let W_0, W_1 and W_2 be the eigenspaces of $\widetilde{A}$ in W corresponding to eigenvalues η_0, η_1 and η_2, respectively. Note that the width of Y denoted as $w(Y)$, i.e., the maximal distance of the vertices of Y in Γ, in our case equals two. Let $\mathbf{1}_Y$ denote the characteristic vector of Y defined by

$$\mathbf{1}_Y = \sum_{y\in Y} \hat{y} \in W.$$

Definition 4.1. A nonzero vector $\boldsymbol{v} \in W$ is said to be *tight* if

$$|\{i \mid i \in \{0, 1, \ldots, D\}, E_i\boldsymbol{v} = \mathbf{0}\}| = w(Y) = 2.$$

Proposition 4.2. *Let $\boldsymbol{v} \in W$ be a nonzero vector such that $E_0\boldsymbol{v} = \mathbf{0}$.*

(i) *For $i \in \{0, 1, \ldots, D\}$,*

$$\frac{\|E_i\boldsymbol{v}\|^2}{\|\boldsymbol{v}\|^2} = \frac{m_i(k-\theta_i)((1+\eta(\boldsymbol{v}))(1+\theta_i)+b_1)}{kb_1|X|} \geq 0, \text{ with } \eta(\boldsymbol{v}) = \frac{{}^t\boldsymbol{v}A\overline{\boldsymbol{v}}}{\|\boldsymbol{v}\|^2}.$$

(ii) *The following hold.*

$$-1 - \frac{b_1}{1+\theta_D} \geq \eta_1 \geq \eta(\boldsymbol{v}) \geq \eta_2 \geq -1 - \frac{b_1}{1+\theta_1}.$$

(iii) *The following are equivalent.*
 (a) *$\boldsymbol{v}$ is tight.*
 (b) *One of the following holds.*
  1. *$\eta(\boldsymbol{v}) = \eta_1 = -1 - \frac{b_1}{1+\theta_D}$ and $\widetilde{A}\boldsymbol{v} = \eta_1\boldsymbol{v}$, or*
  2. *$\eta(\boldsymbol{v}) = \eta_2 = -1 - \frac{b_1}{1+\theta_1}$ and $\widetilde{A}\boldsymbol{v} = \eta_2\boldsymbol{v}$.*

Proof. First observe that $\theta_1 \geq \eta_1 > 0$ and $\theta_D \leq \eta_2 < -1$ by Lemma 4.1(iv).

Since $E_0\boldsymbol{v} = \mathbf{0}$, $\boldsymbol{v}$ can be expressed as a sum of two mutually orthogonal vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in W such that $\widetilde{A}\boldsymbol{v}_1 = \eta_1\boldsymbol{v}_1$ and $\widetilde{A}\boldsymbol{v}_2 = \eta_2\boldsymbol{v}_2$. Hence

$${}^t\boldsymbol{v}A\overline{\boldsymbol{v}} = {}^t\boldsymbol{v}\widetilde{A}\overline{\boldsymbol{v}} = \eta_1\|\boldsymbol{v}_1\|^2 + \eta_2\|\boldsymbol{v}_2\|^2$$

and

$$\eta_1\|\boldsymbol{v}\|^2 \geq \eta_1\|\boldsymbol{v}_1\|^2 + \eta_2\|\boldsymbol{v}_2\|^2 \geq \eta_2\|\boldsymbol{v}\|^2.$$

Thus $\eta_1 \geq \eta(\boldsymbol{v}) \geq \eta_2$ and one of the equalities holds if and only if $\boldsymbol{v}_2 = \mathbf{0}$ or $\boldsymbol{v}_1 = \mathbf{0}$.

The formula in (i) can be computed directly using the fact that $w(Y) = 2$ or see [12, Lemma 11.4]. (ii) is from (i) and the observation above. See also [12, Lemma 11.5].

Since $E_0\boldsymbol{v} = \mathbf{0}$, $\boldsymbol{v}$ is tight if and only if $E_i\boldsymbol{v} = \mathbf{0}$ for some $i \in \{1, 2, \ldots, D\}$. Hence by (i), if $E_i\boldsymbol{v} = \mathbf{0}$, then $i = 1$ or D, and one of the two conditions in (b) holds. The converse is clear. □

Lemma 4.3. *The following hold.*

$$W_1 = \mathrm{Span}(\alpha(u,v) - \eta_2(\hat{u} - \hat{v}) \mid u, v \in Y, \partial(u,v) = 2), \text{ and}$$
$$W_2 = \mathrm{Span}(\alpha(u,v) - \eta_1(\hat{u} - \hat{v}) \mid u, v \in Y, \partial(u,v) = 2),$$

where

$$\alpha(u,v) = \sum_{z \in A(v,u)} \hat{z} - \sum_{w \in A(u,v)} \hat{w}.$$

Proof. It is clear that $W_0 = \mathrm{Span}(\mathbf{1}_Y)$. Hence

$$W_1 + W_2 = W_0^{\perp} \cap W = \mathrm{Span}(\hat{u} - \hat{v} \mid u, v \in Y).$$

We claim that

$$W_1 + W_2 = \mathrm{Span}(\hat{u} - \hat{v} \mid u, v \in Y, \partial(u,v) = 2).$$

Suppose $\partial(u, v) = 1$. Since $\kappa - \mu = a_2 \neq 0$ by Hypothesis 4.1, there exists a vertex $w \in \Gamma_2(u) \cap \Gamma_2(v) \cap Y$. Hence

$$\hat{u} - \hat{v} = (\hat{u} - \hat{w}) + (\hat{w} - \hat{v}).$$

This proves the claim.

It follows from Lemma 4.1 that

$$(\widetilde{A} - \eta_1 I)(\widetilde{A} - \eta_2 I)$$

vanishes on $W_1 + W_2$, and hence

$$W_1 = \mathrm{Span}((\widetilde{A} - \eta_2 I)(\hat{u} - \hat{v}) \mid u, v \in Y \partial(u, v) = 2), \qquad \text{and}$$
$$W_2 = \mathrm{Span}((\widetilde{A} - \eta_1 I)(\hat{u} - \hat{v}) \mid u, v \in Y \partial(u, v) = 2).$$

We now compute $\widetilde{A}(\hat{u} - \hat{v})$ using the fact that Y is strongly closed and that $C(v, u) = C(u, v)$.

$$\begin{aligned}\widetilde{A}(\hat{u} - \hat{v}) &= \widetilde{A}\hat{u} - \widetilde{A}\hat{v} \\ &= \sum_{z \in A(v,u) \cup C(v,u)} \hat{z} - \sum_{w \in A(u,v) \cup C(u,v)} \hat{w} \\ &= \sum_{z \in A(v,u)} \hat{z} - \sum_{w \in A(u,v)} \hat{w} \\ &= \alpha(u, v).\end{aligned}$$

This proves the assertions. □

Proof of Theorem 1.1. The induced subgraph on Y is strongly regular, and

$$\eta_0 = \kappa = c_2 + a_2 > \eta_1 > 0 > -1 > \eta_2$$

by Lemma 4.1. Next we apply Proposition 4.2 to have (2) by (i). Moreover, a nonzero vector $\boldsymbol{v} \in W$ satisfies $E_0 \boldsymbol{v} = E_i \boldsymbol{v} = \boldsymbol{0}$ for some $i \in \{1, 2, \ldots, D\}$ if and only if $\boldsymbol{v}$ is an eigenvector of $\widetilde{A}$ for η_1 or η_2.

Now we have the equivalences in Theorem 1.1(iii) by Lemma 4.3. □

5. Balanced conditions

In this section, we discuss the conditions in Theorem 1.1(ii) from a different point of view by reviewing the balanced conditions defined by Terwilliger in [13,14].

Throughout this section we assume the following hypothesis.

Hypothesis 5.1. Let $\Gamma = (X, R)$ be a distance-regular graph of diameter $D \geq 3$. Let E be a primitive idempotent associated with the eigenvalue $\theta \neq k$ with cosine sequence $1 = \sigma_0, \sigma_1, \ldots, \sigma_D$. Let $V = \boldsymbol{C}^X$, and set $m = \mathrm{rank}\, E$.

By (9), we have

$$E = \frac{m}{|X|} \sum_{i=0}^{D} \sigma_i A_i, \quad AE = \theta E, \tag{12}$$

and for every pair of vertices x, y with $\partial(x, y) = i$,

$$\langle E\hat{x}, E\hat{y}\rangle = \frac{m}{|X|}\sigma_i. \tag{13}$$

Definition 5.1. Let x, y be vertices at distance i. We define vectors $\gamma(x, y)$, $\alpha(x, y)$ and $\beta(x, y) \in V$ by the following.

$$\gamma(x, y) = \gamma_i(x, y) = \sum_{z\in C(y,x)} \hat{z} - \sum_{w\in C(x,y)} \hat{w},$$

$$\alpha(x, y) = \alpha_i(x, y) = \sum_{z\in A(y,x)} \hat{z} - \sum_{w\in A(x,y)} \hat{w}, \qquad \text{and}$$

$$\beta(x, y) = \beta_i(x, y) = \sum_{z\in B(y,x)} \hat{z} - \sum_{w\in B(x,y)} \hat{w}.$$

Definition 5.2. Suppose $\sigma_i \neq 1$.

1. Γ is said to satisfy the $C(i)$-balanced condition if

$$E\gamma(x, y) \in \mathrm{Span}(E(\hat{x} - \hat{y})) \qquad \text{for every } x, y \in X \text{ with } \partial(x, y) = i.$$

2. Γ is said to satisfy the $A(i)$-balanced condition if

$$E\alpha(x, y) \in \mathrm{Span}(E(\hat{x} - \hat{y})) \qquad \text{for every } x, y \in X \text{ with } \partial(x, y) = i.$$

3. Γ is said to satisfy the $B(i)$-balanced condition if

$$E\beta(x, y) \in \mathrm{Span}(E(\hat{x} - \hat{y})) \qquad \text{for every } x, y \in X \text{ with } \partial(x, y) = i.$$

Lemma 5.1. *The following hold.*

(i) $\alpha_1(x, y) = \mathbf{0}$ *if* $\partial(x, y) = 1$, *and* $\gamma_2(x, y) = \mathbf{0}$ *if* $\partial(x, y) = 2$.
(ii) $E(\gamma(x, y) + \alpha(x, y) + \beta(x, y)) = \theta E(\hat{x} - \hat{y})$ *for all pairs of vertices* x *and* y.
(iii) Γ *satisfies the* $B(2)$*-balanced condition if and only if it satisfies the* $A(2)$*-balanced condition.*

Proof. (i) This is clear from Definition 5.1.

(ii) Since E is the primitive idempotent associated with an eigenvalue θ,

$$E(\gamma(x, y) + \alpha(x, y) + \beta(x, y)) = E(A\hat{x} - A\hat{y}) = EA(\hat{x} - \hat{y}) = \theta E(\hat{x} - \hat{y}).$$

(iii) This follows from (i) and (ii). $\square$

By applications of (13) we easily obtain the following.

Lemma 5.2. *Let* u *and* v *be vertices in* Γ *with* $\partial(u, v) = i$. *Then the following hold.*

(i) $\|E(\hat{u} - \hat{v})\|^2 = 2 \cdot \frac{m}{|X|}(\sigma_0 - \sigma_i)$.
(ii) $\langle E\gamma(u, v), E(\hat{u} - \hat{v})\rangle = 2 \cdot c_i \cdot \frac{m}{|X|}(\sigma_1 - \sigma_{i-1})$.
(iii) $\langle E\alpha(u, v), E(\hat{u} - \hat{v})\rangle = 2 \cdot a_i \cdot \frac{m}{|X|}(\sigma_1 - \sigma_i)$.
(iv) $\langle E\beta(u, v), E(\hat{u} - \hat{v})\rangle = 2 \cdot b_i \cdot \frac{m}{|X|}(\sigma_1 - \sigma_{i+1})$.

In [14, Theorem 3.3], Terwilliger proved that Γ is Q-polynomial if and only if Γ satisfies the $B(2)$- and $C(3)$-balanced conditions under the condition that $\sigma_i \neq 1$ for every $i \geq 1$.

Let x and y be vertices with $\partial(x, y) = i$. Let

$$\boldsymbol{u} \in \{E\gamma(x, y), E\alpha(x, y), E\beta(x, y)\} \quad \text{and} \quad \boldsymbol{v} = E(\hat{x} - \hat{y}).$$

Suppose $\sigma_i \neq 1$. Then by Lemma 5.2(i), $\boldsymbol{v} \neq \mathbf{0}$. Now by the Cauchy–Schwartz inequality,

$$\|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2 - |\langle \boldsymbol{u}, \boldsymbol{v}\rangle|^2 \geq 0, \tag{14}$$

and equality holds if and only if $\boldsymbol{u} \in \mathrm{Span}(\boldsymbol{v})$. As in [14], the corresponding balanced conditions can be expressed in terms of the parameters of Γ by taking the sum over all pairs of vertices x, y with $\partial(x, y) = i$. Since $\|\boldsymbol{v}\|^2$ and $\langle \boldsymbol{u}, \boldsymbol{v}\rangle$ are expressed in terms of the parameters of Γ in general by Lemma 5.2, the condition may have a simpler form if the quantity $\|\boldsymbol{u}\|^2$ is expressed only in terms of the parameters of Γ. The following lemma gives formulas when Γ is parallelogram-free.

Lemma 5.3. *Suppose for some integer $i < D$, Γ satisfies the condition SS_j for all j with $1 \leq j \leq i$.*

(i) *Let u and v be vertices in Γ with $\partial(u, v) = i$. Then*

$$\begin{aligned}
&\|E\beta(u, v)\|^2 \\
&\quad = 2 \cdot b_i \cdot \frac{m}{|X|}(\sigma_0 + a_1\sigma_1 + (b_i - a_1 - 1)\sigma_2 \\
&\qquad - (c_{i+1} - c_i)\sigma_i - (a_{i+1} - a_i)\sigma_{i+1} - b_{i+1}\sigma_{i+2}) \\
&\quad = 2 \cdot b_i \cdot \frac{m}{|X|}(\theta(\sigma_1 - \sigma_{i+1}) - c_i(\sigma_2 - \sigma_i) - a_i(\sigma_2 - \sigma_{i+1})).
\end{aligned}$$

(ii) *Let u and v be vertices in Γ with $\partial(u, v) = i + 1$. Then*

$$\begin{aligned}
&\|E\gamma(u, v)\|^2 \\
&\quad = 2 \cdot c_{i+1} \cdot \frac{m}{|X|}(\sigma_0 + (c_{i+1} - 1)\sigma_2 + c_i\sigma_{i-1} - (c_{i+1} - c_i)\sigma_{i+1}).
\end{aligned}$$

Proof. By Corollary 3.3, there is no kite of length $j \in \{2, 3, \ldots, i + 1\}$.

(i) We have

$$\|E\beta(u, v)\|^2 = \left\|\sum_{z \in B(v,u)} E\hat{z}\right\|^2 + \left\|\sum_{w \in B(u,v)} E\hat{w}\right\|^2 - 2\left\langle \sum_{z \in B(v,u)} E\hat{z}, \sum_{w \in B(u,v)} E\hat{w}\right\rangle.$$

Since there is no kite of length 2 or $i + 1$, both $B(v, u)$ and $B(u, v)$ are disjoint unions of $b_i/(a_1 + 1)$ cliques of size $a_1 + 1$. Hence

$$\left\|\sum_{z \in B(v,u)} E\hat{z}\right\|^2 = \left\|\sum_{w \in B(u,v)} E\hat{w}\right\|^2 = b_i \cdot \frac{m}{|X|}(\sigma_0 + a_1\sigma_1 + (b_i - a_1 - 1)\sigma_2).$$

Let $z \in B(v, u)$. Because of the property SS_i, there is no edge between $\Gamma_i(u) \cap \Gamma_i(z)$ and $\Gamma_i(u) \cap \Gamma_{i+1}(z)$. Hence we have the following.

$$|B(u,v) \cap \Gamma_i(z)| = c_{i+1} - c_i,$$
$$|B(u,v) \cap \Gamma_{i+1}(z)| = a_{i+1} - a_i, \qquad \text{and}$$
$$|B(u,v) \cap \Gamma_{i+2}(z)| = b_{i+1}.$$

Therefore we have

$$\left\langle \sum_{z \in B(v,u)} E\hat{z}, \sum_{w \in B(u,v)} E\hat{w} \right\rangle = b_i \cdot \frac{m}{|X|}((c_{i+1} - c_i)\sigma_i + (a_{i+1} - a_i)\sigma_{i+1} + b_{i+1}\sigma_{i+2}).$$

The second equality is obtained by Lemma 2.1(i).

(ii) In this case we have

$$\left\| \sum_{z \in C(v,u)} E\hat{z} \right\|^2 = \left\| \sum_{w \in C(u,v)} E\hat{w} \right\|^2 = c_{i+1} \cdot \frac{m}{|X|}(\sigma_0 + (c_{i+1} - 1)\sigma_2).$$

For $z \in C(v,u)$,

$$|C(u,v) \cap \Gamma_{i-1}(z)| = c_i,$$
$$|C(u,v) \cap \Gamma_i(z)| = 0,$$
$$|C(u,v) \cap \Gamma_{i+1}(z)| = c_{i+1} - c_i.$$

Therefore we obtain the formula. □

Proposition 5.4. *Suppose for some integer $i < D$, Γ satisfies the condition SS_j for all j with $1 \le j \le i$.*

(i) *We have the following.*

$$(\theta(\sigma_1 - \sigma_{i+1}) - c_i(\sigma_2 - \sigma_i) - a_i(\sigma_2 - \sigma_{i+1}))(\sigma_0 - \sigma_i) \ge b_i(\sigma_1 - \sigma_{i+1})^2. \tag{15}$$

$$(\sigma_0 + (c_{i+1} - 1)\sigma_2 + c_i\sigma_{i-1} - (c_{i+1} - c_i)\sigma_{i+1})(\sigma_0 - \sigma_i) \ge c_{i+1}(\sigma_1 - \sigma_i)^2. \tag{16}$$

(ii) *If $\sigma_i \ne 1$, then the following are equivalent.*
   (a) *Γ satisfies the condition $B(i)$.*
   (b) *$E\beta(u,v) \in \mathrm{Span}(E(\hat{u} - \hat{v}))$ for some vertices u, v with $\partial(u,v) = i$.*
   (c) *Equality holds in (15).*

(iii) *If $\sigma_{i+1} \ne 1$, then the following are equivalent.*
   (a) *Γ satisfies the condition $C(i+1)$.*
   (b) *$E\gamma(u,v) \in \mathrm{Span}(E(\hat{u} - \hat{v}))$ for some vertices u, v with $\partial(u,v) = i+1$.*
   (c) *Equality holds in (16).*

Proof. The formulas in (i) are obtained from the Cauchy–Schwartz inequality.

(ii) That (a) implies (b) follows from the definition of the condition. The equivalence of (b) and (c) is from the Cauchy–Schwartz equality condition. (b) and (c) imply (a) as the formula in (15) does not depend on the choices of vertices u, v with $\partial(u,v) = i$.

(iii) This is similar to (ii). □

Proposition 5.5. *Suppose Γ satisfies the conditions SS_1 and SS_2.*

(i) $a_2((b_2-b_1)\theta^2+(2b_2-c_2b_1+a_1b_1)\theta+kb_1-c_2b_1+b_2)\geq 0$.
(ii) *Suppose $a_2\neq 0$. Then the following conditions are equivalent.*
  (a) $(b_2-b_1)\theta^2+(2b_2-c_2b_1+a_1b_1)\theta+kb_1-c_2b_1+b_2=0$.
  (b) *Γ satisfies the condition $B(2)$ with respect to the idempotent associated with θ.*

Proof. (i) We evaluate the quantity below using Lemma 2.1.

$$
\begin{aligned}
&\frac{|X|^2}{4\cdot b_2\cdot m^2}(\|E\beta(u,v)\|^2\|E(\hat{u}-\hat{v})\|^2-|\langle E\beta(u,v),E(\hat{u}-\hat{v})\rangle|^2)\\
&=(\theta(\sigma_1-\sigma_3)-a_2(\sigma_2-\sigma_3))(\sigma_0-\sigma_2)-b_2(\sigma_1-\sigma_3)^2\\
&=\frac{(k-\theta)^2}{k^2b_1^2b_2}(\theta(\theta^2+(k-a_1-a_2)\theta-a_2)-a_2(\theta^2+(c_2-a_1)\theta+c_2-k))\\
&\quad\times(\theta+1+b_1)-(\theta^2+(k-a_1-a_2)\theta-a_2)^2\\
&=\frac{(k-\theta)^2}{k^2b_1^2b_2}((k-a_1-a_2)(1+b_1)-a_2(c_2-a_2+1)\\
&\quad+2a_2-(k-a_1-a_2)^2)\theta^2\\
&\quad+(-a_2(c_2-a_2+1)(1+b_1)-a_2(c_2-k)+2a_2(k-a_1-a_2))\theta\\
&\quad+(-a_2(c_2-k)(1+b_1)-a_2^2)\\
&=a_2\cdot\frac{(k-\theta)^2}{k^2b_1^2b_2}((b_2-b_1)\theta^2+(2b_2-c_2b_1+a_1b_1)\theta+kb_1-c_2b_1+b_2).
\end{aligned}
$$

Now the result follows from Proposition 5.4(i).

(ii) Since $a_2\neq 0$, we have $\theta+1+b_1>0$ by Lemma 2.2. Hence by Lemma 2.1(iv), $\sigma_2\neq 1$. Therefore $E(\hat{u}-\hat{v})\neq\mathbf{0}$ by Lemma 5.2. Now the assertion follows from (i) and Proposition 5.4(ii). □

Proposition 5.6. *Suppose Γ satisfies the conditions SS_1 and SS_2. Assume $\sigma_2\neq 1$. Then the following are equivalent.*

(i) *For a pair of vertices u and v in Γ with $\partial(u,v)=2$,*

$$E\beta(u,v)\in\mathrm{Span}(E\hat{u}-E\hat{v}).$$

(ii) *For every pair of vertices u and v in Γ with $\partial(u,v)=2$,*

$$E\alpha(u,v)=\xi(E\hat{u}-E\hat{v}),\qquad \textit{where } \xi=\frac{a_2(\sigma_1-\sigma_2)}{\sigma_0-\sigma_2}=\frac{a_2(\theta+1)}{\theta+1+b_1}.$$

(iii) $a_2((b_2-b_1)\theta^2+(2b_2-c_2b_1+a_1b_1)\theta+kb_1-c_2b_1+b_2)=0$.

Proof. All assertions follow from Proposition 5.4 except the value of ξ, which can be evaluated by taking the inner products with $E\hat{u}$. □

Corollary 5.7. *Let Γ be a distance-regular graph of diameter $D\geq 3$ with parameters $b_1=b_2$ and $a_2>0$. If Γ satisfies the $B(2)$-balanced condition with respect to a primitive idempotent E_i, then $i=D$ and the eigenvalue associated with E_D is $-k_1/(1+a_1)$.*

Proof. Since $b_1 = b_2$, $c_2 = 1$ by Proposition 5.4.3 in [2]. It is easy to see that Γ satisfies the conditions SS_1 and SS_2. We apply Proposition 5.5 to get $(a_1+1)\theta+k \geq 0$, and equality holds if and only if Γ satisfies the $B(2)$-balanced condition. Since the value $(a_1+1)\theta - k$ is decreasing with θ, equality holds only when $\theta = \theta_D$. □

6. Parallelogram-free distance-regular graphs

In this section we investigate distance-regular graphs $\Gamma = (X, R)$, satisfying the conditions SS_j for all j $(1 \leq j \leq D-1)$. We call Γ a *parallelogram-free* distance-regular graph. Parallelogram-free distance-regular graphs were extensively studied in [9,15–20].

Definition 6.1. Let Γ denote a distance-regular graph with diameter $D \geq 3$. We say Γ has *classical parameters* (D, q, α, β) whenever the intersection numbers are given by

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}\left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \quad (0 \leq i \leq D),$$
$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right) \quad (0 \leq i \leq D),$$

where

$$\begin{bmatrix} j \\ 1 \end{bmatrix} := 1 + q + q^2 + \cdots + q^{j-1}.$$

All distance-regular graphs with classical parameters are Q-polynomial. See [2] for details. The following theorem will be used in the proof of our results.

Theorem 6.1 (*[14, Theorem 4.1]*). *Let Γ denote a distance-regular graph with diameter $D \geq 3$, and let $q \in \boldsymbol{R} \setminus \{0, -1\}$. Then the following conditions* (i), (ii) *are equivalent.*

(i) *Γ has a nontrivial cosine sequence $\sigma_0, \sigma_1, \ldots, \sigma_D$ such that*

$$\sigma_{i-1} - q\sigma_i \text{ is independent of } i \ (1 \leq i \leq D).$$

(ii) *The intersection numbers of Γ are such that*

$$qc_i - b_i - q(qc_{i-1} - b_{i-1}) \text{ is independent of } i \ (1 \leq i \leq D).$$

Furthermore, if (i), (ii) *hold, then*

$$c_3 \geq (c_2 - q)(1 + q + q^2).$$

Theorem 6.2 (*[14, Theorem 4.2]*). *Let Γ denote a distance-regular graph with diameter $D \geq 3$, and let $q \in \boldsymbol{R} \setminus \{0, -1\}$. Then the following conditions* (i), (ii) *are equivalent.*

(i) *Statements* (i), (ii) *hold in Theorem* 6.1, *and*

$$c_3 = (c_2 - q)(1 + q + q^2).$$

(ii) *There exist $\alpha, \beta \in \boldsymbol{R}$ such that Γ has classical parameters (D, q, α, β).*

Lemma 6.3. *Let Γ denote a distance-regular graph with diameter $D \geq 3$, and let $q \in \boldsymbol{R}$. Suppose the intersection numbers of Γ satisfy the following condition.*

$$qc_i - b_i - q(qc_{i-1} - b_{i-1}) \text{ is independent of } i \ (1 \leq i \leq D). \tag{17}$$

Then the following hold.

(i) $(q + 1 + a_1)(q + 1 - c_2) = a_2 - a_1c_2$.
(ii) $(q + 1 + a_1)(q^2 + q + 1 - c_3) = a_3 - a_1c_3$.

Proof. (i) Set the quantity with $i = 1$ equal to the one with $i = 2$ in (17). Then we have

$$q - b_1 + qb_0 = qc_2 - b_2 - q(q - b_1).$$

By (5), we have

$$q - k + 1 + a_1 + qk = qc_2 - k + c_2 + a_2 + qk - q(q + a_1 + 1).$$

By simplifying the equation we have (i).

(ii) Set the quantity with $i = 2$ equal to the one with $i = 3$ in (17). Then we have

$$qc_2 - b_2 - q(qc_1 - b_1) = qc_3 - b_3 - q(qc_2 - b_2).$$

By (5), we have

$$\begin{aligned}&(q + 1)c_2 - k + a_2 - q^2 + qk - q(1 + a_1)\\ &= (q + 1)c_3 - k + a_3 - q^2c_2 + qk - qc_2 - qa_2.\end{aligned}$$

Subtract the right hand side from the left; we have

$$\begin{aligned}0 &= (q + 1)c_2 + a_2 - q^2 - q(1 + a_1) - (q + 1)c_3 - a_3 + q^2c_2 + qc_2 + qa_2\\
&= (q + 1)(q + 1)c_2 - q(q + 1) - qa_1 + a_2(q + 1) - (q + 1)c_3 - a_3\\
&= (q + 1)(q + 1)c_2 - q(q + 1) - qa_1 + a_2(q + 1) - (q + 1)c_3\\
&\quad - a_3 + c_3a_1 - c_3a_1 - c_2a_1(q + 1) + c_2a_1(q + 1)\\
&= (q + 1)(q + 1 + a_1)c_2 - q(q + 1 + a_1) - (q + 1 + a_1)c_3\\
&\quad - (a_3 - c_3a_1) + (a_2 - c_2a_1)(q + 1)\\
&= (q + 1 + a_1)(c_2(q + 1) - q - c_3) - (a_3 - c_3a_1) + (a_2 - c_2a_1)(q + 1)\\
&= (q + 1 + a_1)(c_2(q + 1) - q - c_3 + (q + 1)(q + 1 - c_2)) - (a_3 - c_3a_1)\\
&= (q + 1 + a_1)(q^2 + q + 1 - c_3) - (a_3 - c_3a_1),\end{aligned}$$

as desired. We used the formula in (i). □

We consider the case (ii) in Proposition 5.6 when Γ is a parallelogram-free distance-regular graph with $a_2 > 0$. By Lemmas 2.1 and 2.2, $\sigma_2 \neq 1$.

Lemma 6.4. *Let $\Gamma = (X, R)$ be a parallelogram-free distance-regular graph with diameter $D \geq 3$ and intersection number $a_2 > 0$. Let E be the primitive idempotent associated with an eigenvalue $\theta \neq \theta_0$, and let $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote the corresponding cosine sequence. Let u and v be vertices with $\partial(u, v) = 2$. Suppose*

$$E\alpha(u, v) = \xi(E\hat{u} - E\hat{v}), \text{ where } \xi = \frac{a_2(\sigma_1 - \sigma_2)}{\sigma_0 - \sigma_2} = \frac{a_2(\theta + 1)}{\theta + 1 + b_1}.$$

Then $\theta = \theta_1$ or θ_D, $\xi \neq 0$, and the following hold.

(i) *$\sigma_{i-1} - q\sigma_i$ is independent of i ($1 \le i \le D$), where $q = a_2/\xi - 1$.*
(ii) *If $\theta = \theta_1$, then $q + 1 \ge c_2$ and $q^2 + q + 1 \ge c_3$, and if $\theta = \theta_D$, then $q + 1 \le -a_1$.*

Proof. We first claim that $\theta = \theta_1$ or θ_D. If $b_1 > b_2$, then by Proposition 3.1 there is a strongly closed strongly regular subgraph Y. Since $\alpha(u, v) - \xi(\hat{u} - \hat{v})$ is a tight vector in E^*V, $\theta = \theta_1$ or θ_D by Theorem 1.1. If $b_1 = b_2$, then by Corollary 5.7 we have the claim.

Now we also have $\xi \neq 0$ by Lemma 2.2.

Let $x \in \Gamma_{i-2}(u) \cap \Gamma_i(v)$ for $2 \le i \le D$. Then by Corollary 3.3 the inner product of $E\hat{x}$ with $E\alpha(u, v) = \xi(E\hat{u} - E\hat{v})$ yields

$$a_2(\sigma_{i-1} - \sigma_i) = \xi(\sigma_{i-2} - \sigma_i).$$

Hence with $a_2/\xi = q + 1$ we have

$$(q + 1)(\sigma_{i-1} - \sigma_i) = \sigma_{i-2} - \sigma_i$$

and

$$\sigma_{i-2} - q\sigma_{i-1} = \sigma_{i-1} - q\sigma_i.$$

Therefore we have (i).

(ii) Observe that $a_2 \ge a_1c_2$ and $a_3 \ge a_1c_3$. We apply Lemma 6.3 to have

$$(q + 1 + a_1)(q + 1 - c_2) \ge 0, \qquad \text{and} \qquad (q + 1 + a_1)(q^2 + q + 1 - c_3) \ge 0.$$

Suppose $\theta = \theta_1$. Then $q + 1 + a_1 > 0$. Hence $q + 1 - c_2 \ge 0$ and $q^2 + q + 1 - c_3 \ge 0$. If $\theta = \theta_D$. Then $q + 1 - c_2 < 0$. Hence $q + 1 + a_1 \le 0$. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Suppose $\Gamma = (X, R)$ is a parallelogram-free distance-regular graph with diameter $D \ge 3$, and intersection numbers $a_2 > 0$ and $b_1 > b_2$. Then by Proposition 3.1, Γ has a strongly closed regular subgraph of diameter 2, and Γ satisfies Hypothesis 4.1. Hence we have (i).

Suppose $\theta \in \{\theta_1, \theta_D\}$ attains one of the bounds in (i). Then by Theorem 1.1, the $A(2)$-balanced condition is satisfied. Now we apply Lemma 6.4. Note that

$$q = \frac{a_2}{\xi} - 1 = \frac{\theta + 1 + b_1}{\theta + 1} - 1 = \frac{b_1}{\theta + 1}.$$

Hence we have all assertions. □

Proposition 6.5. *Let $\Gamma = (X, R)$ be a parallelogram-free distance-regular graph with diameter $D \ge 3$ and intersection numbers $a_2 = s - 1 > 0$, $b_1 = b_2$. Suppose the $B(2)$-condition holds with respect to a primitive idempotent E_i. Then Γ is a regular near $2D$-gon and $c_3 \ge 1 - q^3$, where $q = -s = -(a_1 + 1)$. If equality holds, then Γ is a classical distance-regular graph with parameters*

$$(D, q, \alpha, \beta) = \left(D, -s, \frac{s}{1 - s}, \frac{k(1 + s)}{1 - (-s)^D}\right).$$

Proof. We apply Corollary 5.7. Now the number q in Lemma 6.3 is $-s$ as we set $a_1 = a_2 = s - 1$. By Theorem 6.2, Γ is a classical distance-regular graph with parameters $(D, -s, \alpha, \beta)$ for some reals α and β. Now the values of α and β can be computed by Proposition 6.2.1 in [2]. $\square$

7. Examples

1. If Γ contains a strongly closed subgraph isomorphic to (the collinearity graph of) a generalized quadrangle, θ_D attains the second bound in (2) if and only if $\theta_D = -k/(a_1 + 1)$.
2. Dual polar graphs and Hamming graphs are the only Q-polynomial regular near polygons of diameter $D \geq 4$ with intersection numbers $c_2 > 1$ and $a_1 \neq 0$ by Corollary 5.7 in [18] and these are distance-regular graphs having classical parameters with $\alpha = 0$ and $a_1 \neq 0$. These graphs are Q-polynomial with respect to θ_1 and attain both of the bounds in (2).
3. Let Γ be a parallelogram-free Q-polynomial distance-regular graph of diameter $D \geq 4$ with $a_2 \neq 0$. Then Γ has classical parameters (D, q, α, β) and Γ is either a regular near polygon or $q < -1$. Distance-regular graphs having classical parameters (D, q, α, β) with $q < -1$ are said to be of negative type. These graphs satisfy the second bound in (2) [9,15,17–20].

   For a list of negative type distance-regular graphs among others with classical parameters, see [2, Table 6.1].

The author does not know of any parallelogram-free distance-regular graph with intersection number $a_2 \neq 0$ of diameter $D \geq 3$ which is not Q-polynomial.

In a forthcoming paper [7] we study distance-regular graphs with a subset Y such that $E^*V \cap \mathbf{1}_Y^{\perp}$ is spanned by tight vectors.

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